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A new criterion for the existence of KdV solitons in ferromagnets

H Leblond

Laboratoire POMA, UMR-CNRS 6136, Université d'Angers, 2 B^{d} Lavoisier 49045, Angers Cedex 01, France

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Abstract

The long-time evolution of the KdV-type solitons propagating in ferromagnetic materials is considered through a multi-time formalism, it is governed by all equations of the KdV hierarchy. The scaling coefficients of the higher order time variables are explicitly computed in terms of the physical parameters, showing that the KdV asymptotic is valid only when the angle between the propagation direction and the external magnetic field is large enough. The one-soliton solution of the KdV hierarchy is written down in terms of the physical parameters. A maximum value of the soliton parameter is determined, above which the perturbative approach is not valid. Below this value, the KdV soliton conserves its properties during an infinite propagation time.

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1. Introduction

1.1. KdV-type solitons in ferromagnets

Electromagnetic wave propagation in ferromagnetic media is intrinsically nonlinear. It is therefore a matter for intensive research in the theoretical physics of the nonlinear waves. Within the context of the Maxwell–Landau model, analytical expressions describing solitary-wave propagation out from any slowly envelope or long-wave approximation have been found [1]. These waves have also been studied numerically [2]. Envelope solitons have been studied from several theoretical approaches [4, 3]. There are many experiments regarding magnetostatic waves in thin films [5–7]. Long-wave-type approximations allow us to describe some features related to relativistic domain wall propagation [8, 9], but have also brought forward the existence of another type of wave, described by the Korteweg–de Vries (KdV) equation [10]. It has been shown that such a wave can be emitted by a transverse instability of the relativistic domain wall [2, 9]. The interaction between the two types of waves has also been studied [10].

The KdV model is obviously a rough approximation. In [10], where it was first derived in this frame, anisotropy, damping and inhomogeneous exchange were neglected. Second it assumes that the wave depends on a single spatial coordinate (plane wave) and that the amplitude is weak enough, the wavelength and the propagation distance large enough, so that the first order of the KdV approximation can be retained. A study taking into account three space dimensions, damping and inhomogeneous exchange is published independently [11]. But the weakly nonlinear approximation itself may necessitate higher order corrections. The latter are independent of the former ones. Indeed, the wave is intrinsically nonlinear, and the weakly nonlinear approximation is forced by the introduction of a static field to which the wave field can be compared. Even in the roughest approximation, the ratio between the two fields can become rather close to one. A derivation of the equations describing the evolution of the higher order terms has been derived using a multi-time formalism [12]. It allows us to prove that a formal asymptotic expansion exists up to any order, with all its terms bounded [13], which is the first step in the mathematical justification of the convergence of the expansion. Following the idea by Kraenkel *et al* [14, 15], the multi-times expansion for KdV uses the KdV hierarchy. The evolution of the main term in the expansion relative to each higher order time variable is given by the corresponding equation in the KdV hierarchy. Regarding the main term only, all information about the particular physical situation considered is contained in scaling coefficients of the time variables. These coefficients can be computed. The aim of this paper is to give the value of these quantities and to draw physical consequences from them. It is organized as follows: in section 2, we describe the perturbative scheme in the multi-time formalism. In section 3, we compute explicitly the time scaling coefficients. Conclusions can be drawn on the validity of the pertubative scheme considered as an asymptotic expansion, i.e. for a fixed number of terms, when the perturbative parameter becomes small enough. In section 4, we give the expression of the one-soliton solution of the complete KdV hierarchy. This gives information about the validity of the perturbative scheme considered as a series expansion, i.e. for a fixed value of the perturbative parameter and an infinite number of terms.

2. The multi-time formalism

2.1. The KdV mode

The evolution of the magnetization density $\vec{M}$ in a magnetic field $\vec{H}$ is described by the Landau equation

$$\partial_t \vec{M} = -\gamma\mu_0 \vec{M} \wedge \vec{H}_{\text{eff}} \tag{1}$$

where γ is the gyromagnetic ratio ($\gamma > 0$) and μ_0 the magnetic permeability in vacuum. The effective field $\vec{H}_{\text{eff}}$ contains several terms accounting for the inhomogeneous exchange interaction, the effects of finite size and the anisotropy. Here we use the basic approximation: $\vec{H}_{\text{eff}} = \vec{H}$. Damping is also neglected.

The evolution of the magnetic field $\vec{H}$ is described by the Maxwell equations. We assume that, regarding its dielectric properties, the material is perfectly linear and isotropic, and we denote by c the velocity of light based on its dielectric constant $\hat{\varepsilon}$, i.e. $c = 1/\sqrt{\hat{\varepsilon}\mu_0}$. The Maxwell equations then reduce to

$$-\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) + \Delta\vec{H} = \frac{1}{c^2}\partial_t^2(\vec{H}+\vec{M}). \tag{2}$$

We replace below $\vec{H}$, $\vec{M}$ and t by the normalized quantities $\gamma\mu_0\vec{H}/c$, $\gamma\mu_0\vec{M}/c$ and ct. The constants $\gamma\mu_0$ and c then take the value 1.

The 'long-wave' limit of a wave with negative helicity is considered. We introduce a small parameter ε, such that $1/\varepsilon$ measures the length of the solitary wave and ε^2 its amplitude. The magnetic field is expanded as

$$\vec{H} = \vec{H}_0 + \varepsilon^2 \vec{H}_2 + \cdots \tag{3}$$

and $\vec{M}$ in an analogous way. Using the slow variables

$$\begin{cases} \xi = & \varepsilon(x - Vt) \\ \tau_1 = & \varepsilon^3 t \end{cases} \tag{4}$$

it is shown first that this wave propagates at the velocity

$$V = \sqrt{(\alpha + \sin^2\theta)/(\alpha + 1)} \tag{5}$$

where θ is the angle between the propagation direction and the applied field, and $\alpha = H_0/M_0$ the ratio of the latter to the saturation magnetization. Secondly, it is shown that the propagation of this type of 'long wave' is governed by the KdV equation [10]

$$\partial_{\tau_1}\varphi_2 + q\varphi_2\partial_\xi\varphi_2 + r\partial_\xi^3\varphi_2 = 0 \tag{6}$$

where q and r are real constants given by

$$q = \frac{3}{2}\frac{\cos^2\theta\sin^2\theta\sqrt{1+\alpha}}{(\alpha + \sin^2\theta)^{3/2}} \tag{7}$$

and

$$r = \frac{-1}{2m^2}\frac{\cos^4\theta\sqrt{\alpha + \sin^2\theta}}{\sin^2\theta(1+\alpha)^{7/2}}. \tag{8}$$

φ_2 is the wave amplitude, related to the main component $\vec{H}_2$ and $\vec{M}_2$ of the wave magnetic field and the magnetization density through

$$\vec{H}_2 = \varphi_2\vec{h}_1 \qquad \text{and} \qquad \vec{M}_2 = \varphi_2\vec{m}_1 \tag{9}$$

where $\vec{h}_1$ and $\vec{m}_1$ are polarization vectors defined by

$$\vec{h}_1 = m(1+\alpha)\sin\theta \begin{pmatrix} \frac{\sin\theta\cos\theta}{\alpha+\sin^2\theta} \\ 1 \\ 0 \end{pmatrix} \tag{10}$$

and

$$\vec{m}_1 = \frac{m(1+\alpha)\sin\theta\cos\theta}{\alpha + \sin^2\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}. \tag{11}$$

(We use the normalization of [10, 12], introduced for computational convenience.)

2.2. Higher order terms

Going further in the resolution of the perturbative scheme, it is seen that the field component of order j ($j > 2$) can be written

$$\vec{H}_j = \varphi_j\vec{h}_1 + \vec{h}_j^0(\varphi_2, \varphi_3, \ldots, \varphi_{j-1}) \tag{12}$$

where $\vec{h}_j^0$ is an explicit functional of the lower order amplitudes φ_2, φ_3, up to φ_{j-1}, $\vec{h}_1$ the polarization vector given above by (10) and φ_j is a higher order amplitude. φ_j satisfies a linearized KdV equation of the form

$$\partial_{\tau_1}\varphi_j + q\partial_\xi(\varphi_2\varphi_j) + r\partial_\xi^3\varphi_j = \Xi_j(\varphi_2, \varphi_3, \ldots, \varphi_{j-1}) \tag{13}$$

where the right-hand side (rhs) member Ξ_j is an explicit functional of the lower order amplitudes φ_2, φ_3, up to φ_{j-1}. The parity and homogeneity properties of the expansion allow us to prove that half of these equations admit the zero solution, so that φ_j is non-zero for even j only. Note that the inhomogeneous part $\vec{h}_j^0$ of the jth order magnetic field amplitude $\vec{H}_j$ does not vanish for odd j.

We study the long-time propagation by considering the unbounded or secular solutions, and a multi-time expansion. Therefore, we introduce a sequence of slower and slower temporal variables $\tau_1 = \tau, \tau_2, \tau_3, \ldots,$ defined by $\tau_j = \varepsilon^{2j+1}t$. The propagation is governed by all equations of the KdV hierarchy. In particular, the equation giving the evolution of the leading term φ_2 with regard to the first higher order time variable τ_2 is derived as follows (more detail is given in [12]). φ_4 is the amplitude of the first correction to the main term whose amplitude is φ_2. The equation that determines its evolution can be written in the form

$$\partial_{\tau_1}\varphi_4 + q\partial_\xi(\varphi_2\varphi_4) + r\partial_\xi^3\varphi_4 = -\partial_{\tau_2}\varphi_2 - r_2\partial_\xi^5\varphi_2 + \mathcal{O}_2 \tag{14}$$

where $\mathcal{O}_2$ refers to an expression depending on φ_2 without linear term and r_2 is a real coefficient. Some of the functions φ_4 which are solutions of (14) are secular, i.e. grow linearly with the time τ_1. Consider values of the time variable t about $1/\varepsilon^5$. Then the the time variable $\tau_1 = \varepsilon^3 t$ takes values about $1/\varepsilon^2$, and the secular term in φ_4 becomes of order ε^2 instead of ε^4, due to the factor $\tau_1 \propto 1/\varepsilon^2$. For times with this order of magnitude, this correction term must be taken into account in the expression of the main amplitude φ_2. In order to incorporate the correction into the evolution of the main amplitude φ_2 with regard to the second-order time variable τ_2, we impose some condition on the rhs member of equation (14), so that φ_4 remains bounded (or, more exactly, sublinear). The condition to be satisfied is, thus, that equation (14) does not admit any secular solution. Through an explicit computation in the case where φ_2 is the one-soliton solution of KdV, Kodama and Taniuti [16] have noticed that the secular-producing terms are the terms linear with regard to the solution of lowest order φ_2. The secular solutions φ_4 will, thus, vanish if the linear terms vanish from the rhs member of equation (14). To achieve this, we impose that φ_2 satisfies some partial differential equation such that

$$\partial_{\tau_2}\varphi_2 = -r_2\partial_\xi^5\varphi_2 + \mathcal{O}_2. \tag{15}$$

We still need to determine the nonlinear terms of equation (15), represented by $\mathcal{O}_2$. They are not free but imposed by the compatibility condition between the KdV equation (6) and equation (15), which is the Schwartz condition: $\partial_{\tau_1}\partial_{\tau_2}\varphi_2 = \partial_{\tau_2}\partial_{\tau_1}\varphi_2$. Kraenkel, Manna and Pereira [15] have conjectured and checked on many examples that the only equation that possesses the same homogeneity properties as the rhs member of (14), and that satisfies this condition, is the second equation of what is called the KdV hierarchy.

The KdV hierarchy is the following family of equations [17]:

$$\partial_{T_n} v = \partial_X \mathcal{L}^n v \qquad (n \text{ integer}) \tag{16}$$

where $\mathcal{L}$ is a recurrence operator, defined by

$$\mathcal{L} = -\frac{1}{4}\partial_X^2 - v + \frac{1}{2}\int^X \mathrm{d}X(\partial_X v). \tag{17}$$

For $n = 1$, it is the KdV equation with a normalization that differs from that of (6) $\left(q = \frac{3}{2}, r = \frac{1}{4}\right)$. We identify both using the relations

$$v = \frac{q}{6r}\varphi_2 \qquad X = \xi \qquad \text{and} \qquad T_1 = 4r\tau_1. \tag{18}$$

For $n = 2$, the equation of the hierarchy (16) is

$$\partial_{T_2} v = \tfrac{1}{16}\partial_X^5 v + \tfrac{5}{4}(\partial_X v)\partial_X^2 v + \tfrac{5}{8} v\partial_X^3 v + \tfrac{15}{8}v^2\partial_X v. \tag{19}$$

An important property is the existence of the τ Hirota function [17], that is a function of all variables $(X, T_1, T_2, \ldots)$, related to v by

$$v(X, T_1, T_2, \ldots) = 2\partial_X^2 \ln \tau(X, T_1, T_2, \ldots)$$

(avoid any confusion between the τ Hirota function and the time variables τ_j). The existence of τ ensures that a solution v of the system yielded by all equations of the hierarchy exists and, thus, that the Schwartz condition is satisfied at any order. After an adequate choice of the proportionality constant that connects the time variables of order 2, the variable τ_2 of our expansion and the variable T_2 of the hierarchy, we write

$$T_2 = -16 r_2 \tau_2 \tag{20}$$

the evolution equation to be satisfied by φ_2 is

$$\frac{-1}{16 r_2} \partial_{\tau_2} \varphi_2 = \partial_\xi \mathcal{L}^2 \varphi_2. \tag{21}$$

This way, the linear terms have been removed from equation (15). It remains to justify this procedure, which removes all linear terms from the rhs member of the linearized KdV equation, assuming it polynomial with regard to the solution of KdV, ensures that the solution of the linearized equation is bounded [18]. The KdV equation admits an infinite sequence of conserved densities we denote by $\mathcal{A}_j$, an expression of which can be found in [16]. It has been proved in [18] that the secular-producing terms are the terms proportional to $\partial_\xi \mathcal{A}_j$. Further, the relations existing between the conserved densities $\mathcal{A}_j$ and the recurrence operator $\mathcal{L}$ defined by (17), which allow us to write the hierarchy, allow us to show that the procedure by Kraenkel *et al*, initially intended to remove the linear terms, exactly removes all these secular-producing terms.

On the other hand, the rhs member of the linearized KdV equation that governs the evolution of φ_6 involves φ_4, solution of (14). It is thus necessary to see, when a solution of the linearized KdV equation itself is used in the rhs member, which part of it is secular-producing and which part is not. This is not too difficult. Indeed, this solution is given by its expansion on the basis of the squares Φ_k of the Jost functions related to the KdV equation [16, 18], and we have characterized the fact whether a source term is secular-producing or not by some criterion that involves the coefficients of this expansion and their time-dependence. A last point remains to be studied: the dependence of the higher order terms with regard to the higher order times. We shown that it is governed by a linearized KdV hierarchy [13]. Finally, we have been able to justify that the higher order terms are not secular-producing and to prove that the formal expansion contains bounded terms only.

3. Time scales

The generalization of the above procedure to an arbitrary order $n \geqslant 2$ yields the equation

$$\frac{-1}{(-4)^n r_n} \partial_{\tau_n} \varphi_2 = \partial_\xi \mathcal{L}^n \varphi_2 \tag{22}$$

which governs the evolution of the main amplitude φ_2 with regard to the higher order time variable τ_n. $\mathcal{L}$ is defined by the above formula (17). The scaling coefficient r_n is defined by $r_1 = r$ and the recurrence formula

$$r_{n+1} = \sum_{\substack{(\alpha_j)_{1\leqslant j\leqslant n-1}, k\geqslant 0 \\ (\sum_{j=1}^{n-1} 2j\alpha_j)+k=2n+3}} \Xi((\alpha_j)_{1\leqslant j\leqslant n-1}, k) \prod_{j=1}^{n-1} (-r_j)^{\alpha_j}. \tag{23}$$

The sequence of time variables $\tau_1, \tau_2, \tau_3, \ldots$ involved by the multiple time formalism is thus affected by the sequence of scaling coefficients $r_1, r_2, r_3, \ldots$. The equations of the KdV hierarchy are 'universal', not specific to the physical situation considered. The time scaling coefficients thus contain most physical data about the time evolution of the wave. Further,

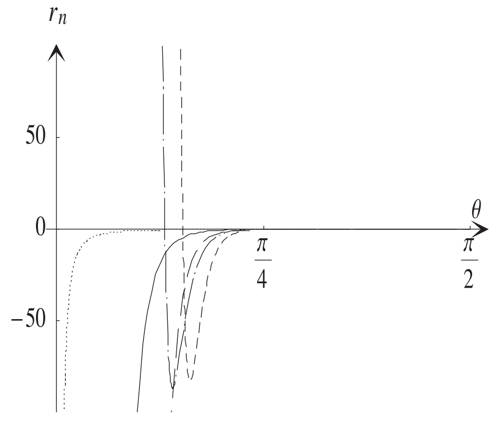

Figure 1. Plot of the five first time scaling coefficients $r_1, \ldots, r_5$ against the angle θ between the propagation direction and the exterior field. The rescaled magnetic induction is $m = 1$, and the parameter determining the strength of the exterior field is $\alpha = 0.5$. Dotted line: r_1, solid line: r_2, long dashed line: r_3, dashed-dotted line: r_4, short dashed line: r_5.

they are of interest regarding the convergence of the asymptotic series. They are computed using recurrence formula (23), together with the results of [12] listed in the appendix. The first coefficients read as follows:

$$r_2 = \frac{\gamma^3 V^9}{8(1+\alpha)^2 m_t^4 \mu}[8 + (4\alpha - 13)\gamma + (3\alpha + 6)\gamma^2 + \alpha(4\alpha - 10)\gamma^3 + 4\alpha(1-\alpha)\gamma^4] \tag{24}$$

$$\begin{aligned} r_3 = \frac{-\gamma^4 V^{13}}{16(1+\alpha)^3 m_t^6 \mu^2} &[40 + (32\alpha - 88)\gamma + (8\alpha^2 + 12\alpha + 67)\gamma^2 + (68\alpha^2 - 132\alpha - 20)\gamma^3 \\ &+ (40\alpha^3 - 143\alpha^2 + 122\alpha + 2)\gamma^4 - \alpha(32\alpha^2 - 70\alpha + 32)\gamma^5 \\ &+ \alpha^2(16\alpha^2 - 56\alpha + 6)\gamma^6 - 8\alpha^2(4\alpha^2 - 8\alpha + 1)\gamma^7 + 16\alpha^3(\alpha - 1)\gamma^8] \end{aligned} \tag{25}$$

in which

$$\gamma = 1 - \frac{1}{V^2} \qquad \mu = 1 + \alpha\gamma \qquad m_t = m \sin\theta. \tag{26}$$

The expressions of the higher order coefficients can be obtained in the same way, but are too complicated to be written down here; numerical computation is more convenient.

The coefficients $r_1(=r)$, $r_2, \ldots$ up to r_5 are plotted in figure 1, against the value of the angle θ between the propagation direction and the external field for a given value of the parameter α that determines the magnitude of this field. Note the annulation and sign change of the coefficients r_4 and r_5, about 0.41 and 0.48 rad, respectively. This marks a change in the behaviour of the corresponding corrections. When r_4 is zero, φ_2 is constant with regard to τ_4, thus the third-order correction is in fact valid at order 4, regarding its time dependence.

It is seen that the r_n take very small values when θ is close to $\pi/2$ and very large values when θ is small. In the limiting case where the propagation direction is orthogonal to the external field ($\theta = \pi/2$), the velocity V is 1, thus $\gamma = 0$, and the coefficients q and r vanish, so that the KdV equation (6) is replaced by

$$\partial_\tau \varphi = 0. \tag{27}$$

Thus φ is constant with time at first order, which means *a priori* that the wave evolves much slower than in the general case, at least for an order of magnitude. Recall that this order of magnitude is determined by the perturbative parameter ε, related to the wave amplitude and typical length. The wave propagates without deformation, to within a quantity of higher order in ε, up to times about T/ε^3, instead of times about T/ε, as usual in the long-wave approximation. The higher order equations simplified by this trivial time evolution of the main term also yield an approximation valid up to T/ε^3 for some finite T. Let us clarify the influence of the scaling coefficients on the time validity range of the higher order approximations. We denote by L_0 some typical length of the wave. The dimensionless space variable is $\xi/L_0 = \varepsilon(x - Vt)/L_0$. The reference length for x is chosen with the order of magnitude of εL_0 in such a way that as x takes values as large as $1/\varepsilon$ with respect to this reference length, ξ is about L_0. Since V is close to 1, it is consistent with the asymptotic expansion to take the same value as a reference time (recall that t has already been rescaled into ct). The higher order time variables adapted to the expansion are, rather than the τ_n, the variables T_n of the KdV hierarchy (22), written under its normalized form

$$\partial_{T_n} v = \partial_\xi \mathcal{L}^n v \qquad (n \text{ integer}). \tag{28}$$

The differential recurrence operator $\mathcal{L}$ is as in (17), with $v = \frac{q}{6r}\varphi_2$. The variable T_n then reads

$$T_n = -(-4)^n r_n \tau_n = -(-4)^n r_n \varepsilon^{2n+1} t. \tag{29}$$

T_n must be about 1 for large values of t. This necessitates a smaller value of ε when the coefficient r_n, or rather $(-4)^n r_n$, is large. For each n, ε must be compared to the reference value ε_n defined as follows: ε_n is the value of ε in (29) such that when $T_0 = \varepsilon t$ is equal to L_0, $|T_n|$ has the same value. It yields

$$\varepsilon_n = \frac{1}{2r_n^{1/2n}}. \tag{30}$$

The approximation involving the first n time variables (for all written terms) is valid for $t \leqslant T/\varepsilon^{2n+1}$, for some finite T with the order of magnitude of the unity in the initial unit $(\varepsilon L_0 v)$. Taking the scaling coefficients into account, the approximation will be valid for $|T_n| \leqslant T$, that is

$$|t| \leqslant \frac{T}{4^n r_n \varepsilon^{2n+1}} = \frac{T}{\varepsilon}\left(\frac{\varepsilon_n}{\varepsilon}\right)^{2n}. \tag{31}$$

Note that it is in fact necessary that $|T_p| \leqslant T$ for all $p \leqslant n$, which implies some conditions on the variations of ε_p in relation to p. According to (31), when ε_n takes large values the propagation can be described over a long distance even if the order n is relatively low and the value of the perturbative parameter ε close to 1. The higher order time variables make sense only if ε is smaller than the ε_n and long time propagation can be described only if the ratio $\varepsilon/\varepsilon_n$ is very small. These conditions will hardly be fulfilled when ε_n becomes small.

The five first ε_n are plotted in figure 2 against the angle θ and in figure 3 against the ratio α that determines the magnitude of the external field. If the extrapolation of the few computed terms is valid, the sequence ε_n seems to be bounded with regard to n, although its terms grow as θ tends to $\pi/2$. Further, this bound is not excessively small when θ is not smaller than 10° or 15°. For smaller values of θ, the ε_n are so small that the KdV approximation can be valid only for excessively low intensities, and the higher orders will never appear.

When θ approaches $\pi/2$, ε_n become large. Then the approximation yielded by the KdV equations is valid for a very long time. The pulse behaviour will be correctly described by them even if the small perturbative parameter ε takes values rather close to 1. At the limit $\theta = \pi/2$, the modulation described by the KdV equation itself arises only at a very slow rate. A typical dependence of the ε_n with regard to the strength of the external field is shown in figure 3. The ε_n grow slowly with α. Thus a strong external field enhances the validity of the

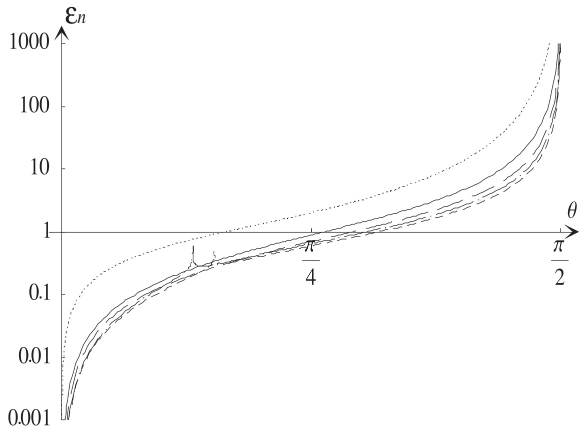

Figure 2. Logarithmic plot of the five first ε_n, reference values for the perturbative parameter ε, built from the time scaling coefficients r_n, against the angle θ between the propagation direction and the exterior field. The values of the constants and the legend are the same as in figure 1.

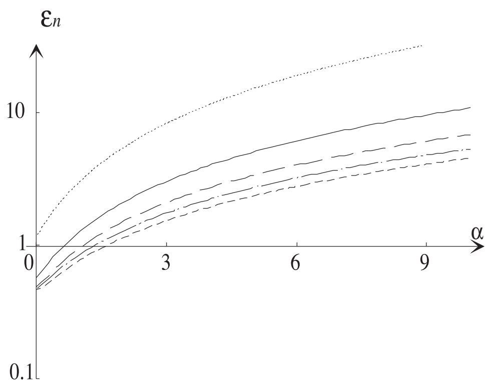

Figure 3. Logarithmic plot of $\varepsilon_1, \ldots, \varepsilon_5$, against the parameter α that determines the strength of the exterior field. The rescaled magnetic induction is $m = 1$, and the angle between the propagation direction and the exterior field is $\theta = \pi/4$. The legend is the same as in figure 1.

KdV approximation and increases the duration for which it can be expected to describe the physics. However this effect is much weaker than the dependence with regard to the direction of the external field and the angle θ.

4. The soliton of the hierarchy

The time scaling coefficients studied in the previous section have given an insight into the convergence of the perturbative expansion as an asymptotic behaviour for small values of the perturbative parameter ε for a fixed number n of corrective terms. We are also able to get some insight into the convergence of the series when n tends to infinity and ε is fixed, through the computation of the one-soliton solution of the complete KdV hierarchy. As mentioned above, all equation of the KdV hierarchy are compatible with each other in the sense that for given initial data, a function $v(X, T_1, T_2, T_3, \ldots)$ satisfying equation (16) for any value of n can be found. This solution can be found using the inverse scattering transform (IST) method, at least in principle. Indeed, all equations of the hierarchy are completely integrable by means of the IST method. Furthermore, they can all be described in the IST formalism using the same spectral problem ([19], p 96), which ensures their compatibility. The scattering data $(R_+(k), D_{+,j}, k_j)$ (see [19], p 141 ff, for the precise definition of these quantities) are defined in the same way for all equations, only their time evolution differs for each time variable T_n. These time evolutions are given by ([19], p 149)

$$R_+(k, T_n) = R_+(k, 0)\, \mathrm{e}^{\Omega_n(k)T_n} \tag{32}$$

$$D_{+,j}(T_n) = D_{+,j}(0)\, \mathrm{e}^{\Omega_{n,j}T_n} \tag{33}$$

$$k_j(T_n) = k_j(0). \tag{34}$$

The index n refers to the nth equation of the hierarchy. The evolution factors are $\Omega_{n,j} = \Omega_n(k_j)$ and $\Omega_n(k) = -\mathrm{i}\omega_n(2k)$, where $\omega_n(k)$ is the dispersion relation of the nth equation of the hierarchy linearized. It is seen from relation (34) that the discrete spectrum (k_j) is constant with regard to any of the time variables T_n. Therefore the number of solitons and their characteristics are not modified by the higher order time evolution. The evolution of the spectral data with regard to all the higher order time variables can then be written as a single exponential factor for each spectral component

$$R_+(k, T_1, T_2, \ldots) = R_+(k, 0, 0, \ldots) \exp\left(\sum_{n=1}^{\infty} \Omega_n(k)T_n\right). \tag{35}$$

From expressions (16) and (17) of the equations of the hierarchy, we find that

$$\omega_n(k) = \frac{-k^{2n+1}}{4^n}. \tag{36}$$

For a value of the spectral parameter k belonging to the discrete spectrum, $k = k_j = \mathrm{i}\kappa_j$ with κ_j real, we get

$$\Omega_{n,j} = 2(-1)^{n+1}\kappa_j^{2n+1}. \tag{37}$$

Using definition (29) of the time variable T_j, we get the following expression of the complete time evolution factor:

$$\sum_{n=1}^{\infty} \Omega_{n,j}T_n = \Omega_j t \qquad \text{with} \qquad \Omega_j = \sum_{n=1}^{\infty} (2\varepsilon\kappa_j)^{2n+1} r_n. \tag{38}$$

Obviously, formula (38) is valid only if the power series converges. Note that the coefficients of the latter are the time scaling coefficients r_n. For a one-soliton solution, the above formulae show that the introduction of a sequence of higher order time variables and of all equations of the KdV hierarchy yield nothing but a renormalization of the soliton speed. This result can be also found by direct computation as follows. By definition, the one-soliton solution propagates without deformation, at least with regard to the first time variable T_1. It can thus be written in the form $v = v(X + \lambda T_1)$. Then using the KdV equation, i.e. equation (16) with $n = 1$, we see that v is an eigenvector of the recurrence operator $\mathcal{L}$ defined by (17), with the eigenvalue λ. We deduce easily the T_n-dependence of v, it is given by $v = v(X + \lambda^n T_n)$. We find in this way, the expression of the one-soliton solution of the complete hierarchy

$$v = 2b^2 \operatorname{sech}^2 b\left(X + \sum_{n=1}^{\infty} (-b^2)^n T_n\right) \tag{39}$$

using the normalized variables. In the case of magnetic solitons, it can be written using the physical variables as

$$\vec{H}_w = \frac{12r}{q}\vec{h}_1\beta^2 \operatorname{sech}^2 \beta(x - \mathcal{V}t) \tag{40}$$

where

$$\mathcal{V} = V + \sum_{n=1}^{\infty} 4^n \beta^{2n} r_n. \tag{41}$$

V is the velocity given by (5), $\vec{h}_1$ the polarization vector given by (10). The wave magnetic field $\vec{H}_w$ is related to the previously defined field components through

$$\vec{H} = \vec{H}_0 + \varepsilon^2 \vec{H}_2 + \cdots \simeq \vec{H}_0 + \vec{H}_w. \tag{42}$$

The dimensional soliton parameter β is related to the normalized soliton parameter b through $\beta = \varepsilon b$. Computation of the one-soliton from the IST formalism allows us to identify the soliton parameter b with the single discrete eigenvalue κ_1. This way, we check that the relative soliton velocity $(\mathcal{V} - V)$ given by (41) is equal to $\Omega_1/(2\beta)$, using expression (38) of the evolution factor Ω_1.

The soliton speed is thus given by a power series of the soliton parameter β, whose coefficients are essentially the time scaling coefficients $(r_n)_{n\geqslant 1}$. Obviously, if this series diverges, so does the whole perturbative scheme. Reciprocally, the convergence of the series defining the velocity should favour that of the perturbative scheme, although the latter is by no means proved. Writing the power series which defines $\mathcal{V}$ as

$$\mathcal{V} = V + \sum_{n=1}^{\infty} \left(\frac{\beta}{\varepsilon_n}\right)^{2n} \tag{43}$$

we see that it converges when

$$\beta < \beta_M = \liminf_{n\longrightarrow\infty} \varepsilon_n \tag{44}$$

and diverges for larger values of the soliton parameter β. Therefore the limit of the sequence ε_n for large n gives us a maximal value β_M of the soliton parameter β, above which we know that the perturbative scheme does not converge as the number of terms is increased. Physically, this lack of convergence means that the KdV soliton will be destroyed by some effects which cannot be taken into account using the perturbative approach.

For values of the soliton parameter β below the limit β_M, we get a renormalized soliton speed, *a priori* valid for an infinite propagation time. The boundedness of all terms in the perturbative scheme proves that for a given propagation time and a given number of terms, this soliton gives a good approximation of the real impulsion for small enough values of ε, i.e. of β. We can reasonably conjecture that 'small enough' can be understood here as less than the limiting value β_M of β. Physically it means that magnetic KdV solitons with parameter smaller than β_M should conserve their properties during a long propagation time.

According to figures 2 and 3, the ε_n, and thus also their limit β_M, depend on the physical parameters and especially on the angle θ between the propagation direction and the applied field. An example of computation showing the convergence of the velocity series is drawn in figure 4 as a function of this angle. It is seen that, for θ close to $\pi/2$, the first approximation (KdV) gives almost the exact speed, while for small angles the series diverges. We denote by θ_M the value of θ for which β_M is equal to the fixed value of β. When $\theta < \theta_M$, the series does not converge and the whole perturbative approach is invalid. To compute θ_M for figure 4, we have approximated β_M by ε_5. When $\theta > \theta_M$, if we consider only the soliton speed, the KdV approximation will correctly describe the wave evolution. More precisely, the KdV equation itself will give an acceptable description above some value θ_t of the angle θ, while this first order approximation needs to be corrected by higher order terms below θ_t (note that the threshold value θ_M is precisely defined, while θ_t is only an order of magnitude depending on the accuracy required). It is reasonable to think that the same kind of conclusion holds in a more general situation, involving several solitons and radiation.

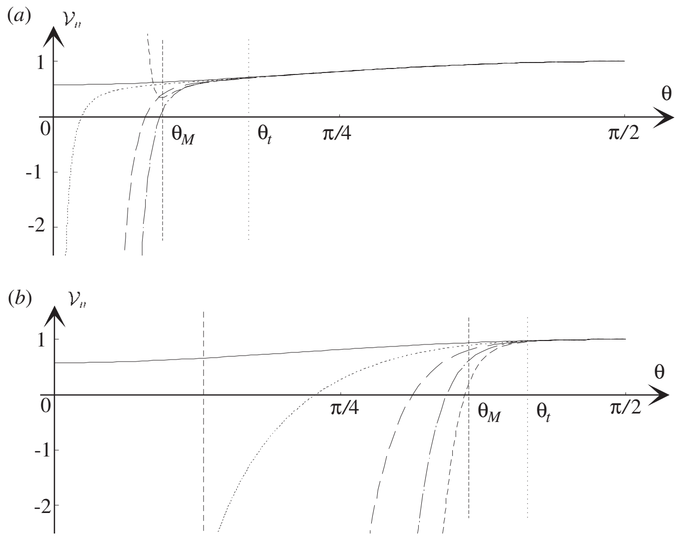

Figure 4. Plot of the five first approximate values $\mathcal{V}_n = V + \sum_{p=1}^{n} (\beta/\varepsilon_n)^{2n}$ of the soliton velocity $\mathcal{V}$, against the angle θ between the propagation direction and the exterior field. The rescaled magnetic induction is $m = 1$ and the parameter determining the strength of the exterior field is $\alpha = 0.5$. Solid line: $n = 0$, dotted line: $n = 1$, long dashed line: $n = 2$, dashed-dotted line: $n = 3$, short dashed line: $n = 4$. For a soliton parameter $\beta = 0.1$ (*a*), $\beta = 1.5$ (*b*).

5. Conclusion

The multiple time formalism has been applied to the study of the propagation of KdV solitons in ferromagnetic media. According to this formalism, the dependence of the higher order terms with respect to the first-order time variable is given by linearized KdV equations, while the dependence of the main term with regard to the higher order time variables is governed by all equations of the KdV hierarchy. The latter are determined by the requirement that the linear terms in the rhs of the linearized KdV equation vanish. This yields scaling coefficients for the higher order time variables of the KdV hierarchy, which contain most physical information concerning the wave evolution. Explicit computation of these scaling coefficients shows, in particular, that the approximation yielded by the KdV model gives a good account of the physical behaviour of the wave during long propagation times when the angle between the propagation direction and the external field is large enough. The time during which the pulse is correctly described by the KdV equation falls to zero when they are parallel. Mathematically, the perturbative parameter ε is infinitely small, while it takes a finite value in a physical situation. The approximation is valid only if this finite value is small enough. The corresponding range of the perturbative parameter ε is usually determined in a rather empirical way. The present study gives some theoretical insight into this question, through a physical interpretation of the time scaling coefficients.

The one-soliton solution of the complete KdV hierarchy has been written down as a function of the physical parameters. The soliton velocity is written as a power series of the soliton parameter, involving the sequence of the time scaling coefficients. We get a maximum value of the soliton parameter, above which the perturbative series diverges. Then the KdV approximation, even with corrective terms, does not describe the physics correctly. If the soliton parameter is below the threshold, the long-distance effect of the higher order corrections is only a modification of the soliton speed, and the physical system behaves qualitatively as the KdV model. It has been observed that the validity domain of KdV-type asymptotics is often much larger than predicted by the mathematical analysis. The above conclusions can partially explain this observation: the KdV-type behaviour is qualitatively correct in the whole validity domain of the infinite KdV hierarchy expansion, which is expected to be much larger.

Appendix

We list in this appendix the formulae needed for the computation of the time scaling coefficients r_n. These formulae are proved in [12]. r_n is given by equation (23) with

$$\Xi((\alpha_j)_{j\geqslant 1},k)=\frac{-1}{\Lambda}\left[V\vec{m}\cdot\vec{m}((\alpha_j)_{j\geqslant 1},k-1)-\sum_{i\geqslant 1}\vec{m}\cdot\vec{m}((\alpha_j-\delta_{i,j})_{j\geqslant 1},k-1)\right] \tag{45}$$

where

$$\vec{m}=\begin{pmatrix}m_x\\ m_t\\ 0\end{pmatrix}. \tag{46}$$

$\vec{m}((\alpha_j),k)$ is deduced according to

$$\tilde{u}((\alpha_j)_{j\geqslant 1},k,l)=\begin{pmatrix}\vec{e}((\alpha_j)_{j\geqslant 1},k,l)\\ \vec{h}((\alpha_j)_{j\geqslant 1},k,l)\\ \vec{m}((\alpha_j)_{j\geqslant 1},k,l)\end{pmatrix} \tag{47}$$

from the following recurrence formulae. For all $l\geqslant 1$

$$\tilde{u}((0),0,l)=\tilde{u}_1. \tag{48}$$

For all k and $l\geqslant 1$

$$\tilde{u}((0),k,l)=S(V\vec{m}((0),k-1,l)). \tag{49}$$

For all $(\alpha_j)_{j\geqslant 1}\neq(0)$ and $l\geqslant 1$

$$\tilde{u}((\alpha_j)_{j\geqslant 1},0,l)=\sum_{i\geqslant 1}\Phi(\tilde{u}((\alpha_j-\delta_{i,j})_{j\geqslant 1},0,l)). \tag{50}$$

For all $(\alpha_j)_{j\geqslant 1}\neq(0),k,l\geqslant 0$

$$\tilde{u}((\alpha_j)_{j\geqslant 1},k,l)=S(V\vec{m}((\alpha_j)_{j\geqslant 1},k-1,l))+\sum_{i\geqslant 1}[\Phi(\tilde{u}((\alpha_j-\delta_{i,j})_{j\geqslant 1},0,l)) - S(\vec{m}((\alpha_j-\delta_{i,j})_{j\geqslant 1},k-1,l))]. \tag{51}$$

In (48)–(51), $\delta_{i,j}$ is the Kronecker symbol, V is given by (5). Φ is defined by

$$\Phi(u)=S(\alpha\vec{m}\wedge\vec{m}_I(u))+u_I(u) \tag{52}$$

where S is the 9×3 matrix

$$S=TL^{-1}\qquad\text{with}\quad T=\begin{pmatrix}-\frac{1}{V}R_x\\ I\\ -\Gamma\end{pmatrix}. \tag{53}$$

I is the three-dimensional unity matrix, and

$$L^{-1}=\frac{1}{\mu m_x m_t}\begin{pmatrix}0&0&0\\ 0&0&m_t\\ m_x&0&0\end{pmatrix}. \tag{54}$$

u_I is the linear operator in $\mathbb{R}^9$ defined by

$$u_I\left(\begin{pmatrix}\vec{E}\\ \vec{H}\\ \vec{M}\end{pmatrix}\right)=\begin{pmatrix}\frac{1}{V}\vec{E}\\ 0\\ \vec{m}_I\left(\begin{pmatrix}\vec{E}\\ \vec{H}\\ \vec{M}\end{pmatrix}\right)\end{pmatrix} \tag{55}$$

with

$$\vec{m}_I\left(\begin{pmatrix}\vec{E}\\ \vec{H}\\ \vec{M}\end{pmatrix}\right)=\frac{1}{V}(\vec{H}+\vec{M})+\frac{1}{V^2}R_x\vec{E}. \tag{56}$$

R_x is the 3×3 matrix

$$R_x=\begin{pmatrix}0&0&0\\ 0&0&-1\\ 0&1&0\end{pmatrix}. \tag{57}$$

The first term of the sequence $\tilde{u}((\alpha_j)_{j\geqslant 1},k,l)$ is given by $\tilde{u}_1 = T\vec{h}_1$, where $\vec{h}_1$ is the polarization vector defined by (10), which also reads

$$\vec{h}_1=\begin{pmatrix}\mu m_x\\ (1+\alpha)m_t\\ 0\end{pmatrix}. \tag{58}$$

The quantity Λ in (45) is given by

$$\Lambda = V\vec{m}\cdot\vec{\Phi}_m(\tilde{u}_1) \tag{59}$$

where Φ_m is the m-component of Φ defined by (52), according to

$$\Phi=\begin{pmatrix}\vec{\Phi}_e\\ \vec{\Phi}_h\\ \vec{\Phi}_m\end{pmatrix}. \tag{60}$$

We use the shortcuts

$$\Gamma=\begin{pmatrix}1&0&0\\ 0&\gamma&0\\ 0&0&\gamma\end{pmatrix}\qquad m_x = m\cos\theta \tag{61}$$

and γ, μ, m_t given by (26). The expression (5) of the velocity yields the relation

$$\mu m_x^2+\gamma(1+\alpha)m_t^2=0 \tag{62}$$

which is useful to simplify the expressions.

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